M102. Transformations of energy in the motion of a pendulum

Aim:
- *Determination of the real motion of a model of mathematical pendulum*
- *Measurement of time dependence of the pendulum position $\alpha(t)$,*
- *On the basis of $\alpha(t)$calculation of time dependencies of the pendulum velocity v(t),kinetic energy $E_k(t)$,potential energy $E_p(t)$, total energy $E_c(t)$,*
- *Measurement of the period T of pendulum oscillations as a function of the amplitude of oscillations α_0,*
- *Numerical calculation of the dependence $T(\alpha_0)$ and its comparison with experimental one.*

Problems:
- *Harmonic motion,*
- *Equation of the mathematical pendulum in the approximation of small angles,*
- *Newton's second law of dynamics.*

References:
- *HenrykSzydłowski " PracowniaFizyczna", Wyd. Nauk. PWN,Warszawa1994*
- *David Halliday, Robert Resnick, Jearl Walker "Podstawyfizyki" Wyd. Nauk.PWN,Warszawa2003*

1. Introduction

Mathematical pendulum (Fig. 1) is a mass *m* concentrated at a point and hanged on a weightless string of the length *l*, performing periodical oscillations about the point of equilibrium. The oscillations are performed without air resistance and without friction at the point of attachment. Such a pendulum is an idealised representation of a real physical pendulum.

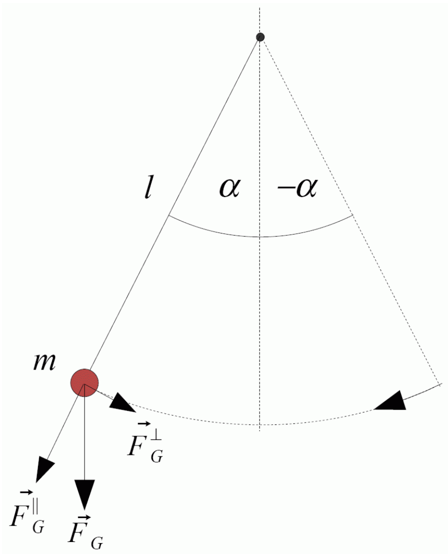


Figure 1. Model of a mathematical pendulum with the forces acting on the mass.

Analysis of the pendulum oscillations is based on the Newton's second law of dynamics. The driving force is the component $\vec{F}_G^{\perp}$ of the gravitation force $\vec{F}_G$, tangent to the arc over which the mass *m* moves. The value of the force is given by

$$\vec{F}_G^{\perp} = -F_G \sin(\alpha) = -mg \sin(\alpha) \,, \tag{1}$$

where *g* is the gravitational constant. According to the Newton's second law of dynamics, the force is known to produce a motion with instantaneous acceleration *a* meeting the relation,

$$\vec{F}_G^{\perp} = ma \;. \tag{2}$$

Taking into account the direction of the force, the kinematic definition of acceleration $a = d^2s/dt^2$(where *s* is the displacement) and the relation between the length of the arc, *s*, and the amplitude of the pendulum, α, $s = l\alpha$, we finally have,

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l}\sin(\alpha) = 0 \,. \tag{3}$$

Equation (3) does not have a simple analytical solution. Only on condition that the amplitude of the pendulum is small so that the relation $\sin(\alpha) = \alpha$ is satisfied, we get a well-known in physics equation for the harmonic oscillations

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l}\alpha = 0 \,. \tag{4}$$

whose solution is,

$$\alpha(t) = \alpha_0 \cos(\omega_0 t + \delta) \tag{5}$$

where α_0 is the amplitude of oscillations, $\omega_0 = \sqrt{g/l}$ is the frequency of normal oscillationsof the pendulum and δ is the initial phase of the movement for $t= 0$. Knowing the frequency of oscillations it is possible to calculate the period of a pendulum in the approximation of small amplitude, from the following equation,

$$T = 2\pi\sqrt{l/g} \tag{6}$$

In general, replacement of eq.(3) by eq.(4) is not justified, which means that the pendulum oscillations are not harmonic and the period of pendulum motion is a function of amplitude.

A simple numerical integration permits finding the period of oscillations for a specific amplitude α_0.

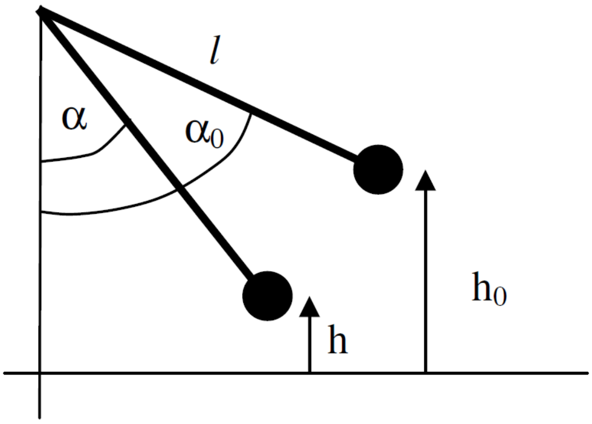


Figure 2. Illustration for numerical determination of the pendulum period.

Increase in the kinetic energy at the transition from α_0 to α is equal to the potential energy difference at these positions, $mg(h_0 - h)$. Assuming α_0 as the starting point ($v = 0$), we have

$$\frac{mv^2(\alpha)}{2} = mg(h_0 - h) \tag{7}$$

And, because $h(\alpha) = l[1 - \cos(\alpha)]$, where $l$is the length of the pendulum,

$$\frac{mv^2(\alpha)}{2} = mgl(\cos(\alpha) - \cos(\alpha_0)) \tag{8}$$

thus

$$v(\alpha) = \sqrt{2gl(\cos(\alpha) - \cos(\alpha_0))} \tag{9}$$

The time needed for covering the arc section $ds = l d\alpha$ is $dt = ds/v$. Integration of dtover the angle α_0 to 0 gives one fourth of the pendulum period:

$$\frac{T}{4} = \int_{\alpha_0}^{0} \frac{l d\alpha}{\sqrt{2gl(\cos(\alpha) - \cos(\alpha_0))}} \,. \tag{10}$$

Simple procedures for numerical integration are given below.

The aim of this experimentis todetermine the dependence of the period of the pendulum oscillations, T, on their amplitude, α_0.Numerical integration of eq. (10) permits the calculation of theoretical value of the pendulum period and its dependence on the oscillations amplitude. Thus, the experimentally determined $T(\alpha_0)$ dependence can be compared with the simulated one.On the basis of numerical calculations you will be able to find the instantaneous values of velocity, kinetic energy, potential energy and total energy.

In our experiment, the mathematical pendulum is approximated by a relatively heavy cylinder (~0.5kg) set on a light aluminium rod of about 1 m in length. Large inertia of the pendulum, its hanging on a high quality ball bearing and small mechanical resistance of the potentiometer used as a sensor of position make such a system a reasonable model of a mathematical pendulum.

Tasks

- *Measure the length of the pendulum and its mass.*
- *Calculate the period of pendulum from formula (6).*

2. Measuring setup.

The quantity to be directly measured is voltage Umeasured on a potentiometer whose axis of rotation coincides with the axis of the pendulum rotation. In order todetermine the real angle of pendulum deviation from the position of equilibrium, corresponding to the voltage U,you should measure the voltage at the position of equilibrium(U_0) and the voltage U_β for a large known amplitude β, e.g. when the pendulum reaches the restricting boundary. Then, the voltage measured U can be easily converted into the amplitude α:

$$\alpha = \beta \frac{U - U_0}{U_\beta - U_0} \tag{11}$$

The voltage is measured with the use of an analog-digital converter card connected to a computer. The communication between the computer and the card is realized through the available set of subroutines.

3. Solution of differential equations of motion(numerical integration)

The problem is to solve the equation of motion on the basis of the known acting force. We want to know current values of the velocity and position of the pendulum on the basis of the instantaneous acceleration calculated from the Newton's second law of dynamics, as the ratio of the instantaneous force to mass.

$$f(t) \approx \frac{F(t+\Delta t) - F(t)}{\Delta t} \tag{12}$$

$$F(t+\Delta t) \approx F(t) + f(t)\Delta t \tag{13}$$

Following the procedure described by equations (12) and (13), you should calculate the position x as a function of time at first for the pair acceleration (f=a)and velocity(F=v) and then for the pair velocity ($f = v$), and position ($F = x$).By choosing a suitably small value of Δt, it is possible to get a good approximation of the real movement of the pendulum. It should be remembered that the driving force of the pendulum is also a function of time, through the dependence on the angle α, eq.(1).

Tasks
Write a program inLabViewto simulate the motion of a mathematical pendulum. As initial parameters use l,g,$\Delta t$and $\alpha 0$. You should get as a result a quantity of Waveformtype containing the information on the pendulum positions at subsequent moments of time $t_i = i\Delta t$.When writing the equations remember about the signs representing the sense of particular quantities, e.g. acceleration is always directed in the opposite to the direction of deviation from the position of equilibrium. .

4. Numerical differentiation

Use again the approximation described by eq. (12),but this time the function $F(t)$ is known and $f(t)$should be found directly from this equation. In this experiment numerical differentiation is used for determination of instantaneous values of the pendulum velocity.

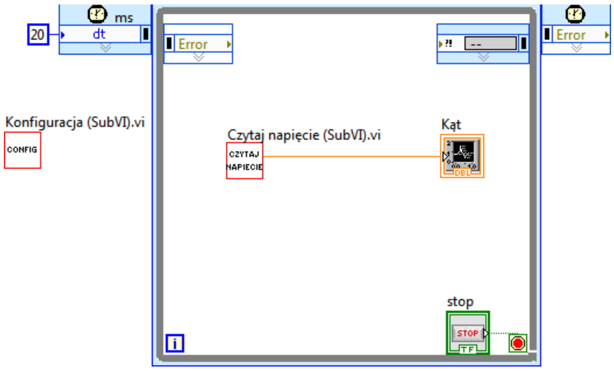
5. Experiment

Write program or programs according to the scheme given below.

The subVI needed for communication with the measuring device are given in the palette "Functions->User Libraries->Pendulum" .

1. Write a program for continuous reading the voltage drop on the potentiometer mounted at the pendulum rod attachment. Use subVI"Configure (subVI)" and"Read voltage (SubVI)'. Use the "Timed Loop"While to perform synchronous voltage reading at the rate of 50values per second (delay of 20ms). The values measured display in the WaveForm Chart

.



2. Use the measured values of voltage and convert them into the angle, voltage [V] into angle [rad]. Using the program written, measure the voltage U_0 corresponding to the equilibrium position of the pendulum. This value corresponds to $\alpha=0$.

   b) Measure the voltage corresponding to the pendulum deviation by 90 to the left (U_{+90}) and to the right (U_{-90}), for the range of 180stopni=π. Find the calibration constant from the equation $C[rad/V] = \dfrac{\pi}{|U_{+90} - U_{-90}|}$

3. Use the function"Normalise (SubVI)"to convert continuously the values of voltage into angles of amplitude. Send the value U_0 to the connector "Zero shift[V]",while the value Cto the connector "Scaling constant[rad/V]".

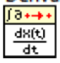
Mechanical energy relations in pendulum motion

1. Make the function (SubVI),to calculate the potential energy of the pendulum on the basis of read off values of the amplitude angles $E_p = mgL(1-\cos(\alpha))$
2. Make the function (SubVI),to calculate the kinetic energy of the pendulum on the basis of the current values of angular velocity,

   $$E_k = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\omega^2 = \frac{1}{2}mL^2(\frac{d\alpha}{dt})^2$$

   To calculate the derivative use the function “Derivativex(t)PtbyPt.vi”to calculate the derivative point by point.

   Derivative x(t) PtByPt.vi

   Besides the current value of the amplitude angle, to calculate dα, the function needs the value of *dt*. When calculating the derivative numerically, *dt*is Δtso the time distance between two measurements of the angle, in the example presented 20ms)
3. Find out the total energy as a sum of kinetic and potential energy terms. Draw the time dependencies of all three terms. Analyse the relationship between the energies, potential energy is exchanged into kinetic energy. The total energy is constant for small time intervals. Over longer time intervals you will note the effect of damping. Over a longer time interval the effect of damping is well seen. This effect is related to the energy dissipation in the friction of the bearing, the potentiometer and the resistance of air. Try to guess the character of the time dependence of the energy dissipation, is it linear or perhaps exponential?

Dependence of the pendulum period on its amplitude –T(α_0)

1. Record the time changes in the pendulum amplitude, starting from high amplitudes to almost total disappearance of oscillations. Write the values of times and angles in a table and then present them as a plot of α(t) in XY Graph.
2. Take advantage of XY Graph cursors tofind out the period of oscillations for a given amplitude (angle). In order to do this, find the positions of the minima or maxima of amplitude.
3. Write a new program (VI). Write the read off cursor positions (the values of *T*andα_0) into the one-dimensional tables. Make an XY Graph of $T(\alpha_0)$.Comment on the result obtained taking

into account its comparison with the result obtained for a mathematical pendulum($T(\alpha_0)$=const.), $T = 2\pi\sqrt{l/g}$.

4. Estimate the value of period that would be observed for $\alpha_0 \rightarrow 0$. Using this value and the equation for the pendulum period, calculate the length of the pendulum. Compare the value obtained with the real length of the pendulum. Comment on the result, answer which value is higher and why.

Additional tasks

1. Read the additional materials provided.
2. Determine the relation $T(\alpha_0)$ from numerical integration of full (not simplified) equation of motion (3). Compare the results with experimental data.
3. Estimate the error following from the approximation of the real pendulum by a mathematical pendulum. Calculate the period of the real pendulum treating it as a physical pendulum. For the sake of simplicity assume that the weight has a ball shape. Weight the aluminium rod.

Technical remarks

Numerical integration

The content given below is given just for illustration. The methods described do not ensure the optimum precision or rate of execution. Those interested should consult any handbook on numerical methods. .

a) Definite integral in finite limits. The method of trapeziums.

The geometrical interpretation of a definite integral $F = \int_a^b f(x)dx$ is the area under the function to be integrated. The interval of integration <a,b> is divided into n parts of the same size, at the points $x_1, x_2, x_3, \ldots, x_{n+1}$ of the width $\Delta x=(b-a)/n$. The area P_i under the plot of the function $f(x)$ in the section (x_i, x_{i+1}) can be approximated by the area of a trapezium of the sides $f(x_i)$ and $f(x_{i+1})$ and the height of Δx, so $P_i = [f(x_i) + f(x_{i+1})]\Delta x/2$. This implies the following equation

$$F = \left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i) \right) \frac{\Delta x}{2} .$$

In LabViewyou can use the function “Numerical Integration”,whose arguments are the values of the integrated function given in a table and the interval Δx.The interval Δxshould be chosen taking into account the rate of the function changes. It is recommended to continue the calculations for decreasing value of Δx as long as it stops showing significant changes.

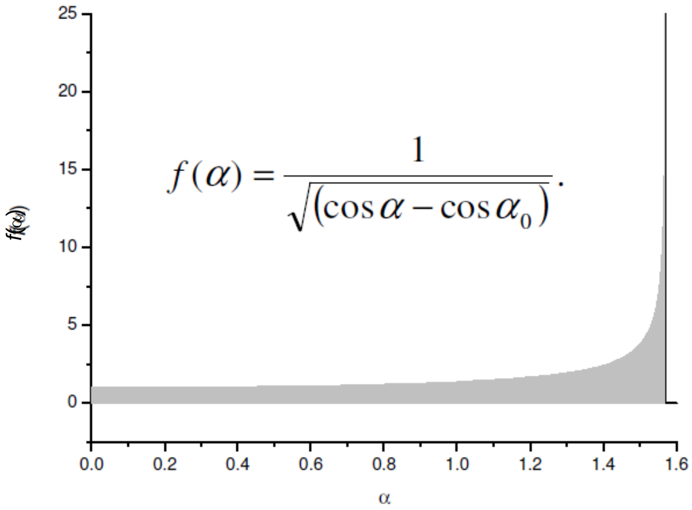


Figure 3. The plot of a fragment of the integrand from eq. (10). The integral is equal to the area marked in grey.

Unfortunately, the integrand from eq. (10) is discontinuous for $\alpha = \alpha_0$(seeFig.3).As the integral is finite despite this, the period of the pendulum has a defined value, the solution is to reduce the value of Δx and approaching α_0 to such a point that the integral value stops changing significantly. As no increase in the density of division is needed in the range through which the integrand is relatively flat, it is recommended to divide the integral into a sum of two integrals calculated for different Δx, different density of division. As the point of division we suggest 90 % of the range of integration. It can be mathematically presented as follows:

$$F = \sum_{j=1}^{N} F_j = \sum_{j=1}^{N} \int_{a_j}^{b_j} f(\alpha)d\alpha$$

where

$$a_1 = a, \qquad b_j = b - \frac{b-a}{10^j}, \qquad a_{j+1} = b_j.$$

For illustration we give a few values of a_j, b_j for$a = 0$ and$b = 1$:

j	1	2	3	4	5	6
a_j	0	0.9	0.99	0.999	0.9999	0.99999
b_j	0.9	0.99	0.999	0.9999	0.99999	0.999999

Assuming a division of each section into n = 100 points, we should get a good accuracy of the integral for N=8,so after 8x100=8000steps.Note that for the density of division from the last section extended over the entire range of integration it is necessary to perform 10^8 steps.

Tasks

Write a program inLabViewto calculate the period of a mathematical pendulum as a function of amplitude of oscillations. After selection of the optimum values of n and N express it in the form of sub-Vi with inputs l, g, α_0*and output giving T.*